

SUPER CONVERGENCE OF CONFORMING FINITE ELEMENT APPROXIMATION FOR THE SECOND ORDER ELLIPTIC PROBLEM WITH ROBIN BOUNDARY CONDITION BY L^2 - PROJECTION METHODS

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ABSTRACT

A general superconvergence result of conforming finite element method for the elliptic problem is established by using L^2 -projection method. Regularity assumption for the elliptic problem with regular partitions are required. Numerical experiment are given to verify the theoretical results.

KEYWORDS: Finite Element Methods, Superconvergence, L^2 -Projection, Elliptic Problem with Robin Boundary Condition

1. INTRODUCTION

The finite element method (FEM) is a numerical technique for solving problems which are described by partial differential equation or can be formulated as functional minimization. To summarize in general terms how FEM works we list main steps of the finite element solution procedure below. The first step is to divide a solution region in to finite elements. The finite element mesh is typically generated by a preprocess or program. The description of mesh consists of several arrays main of which are nodal coordinates and element connectivities. The second step selection interpolation function. The interpolation functions are used to interpolate the field variables over the element. Often polynomial depends on the number of nodes assigned to the element. The third step is to find the element properties. The matrix equation for the finite element should be established which relates the nodal values of the unknown function to other parameters. The fourth step to find the global equation for the whole solution region we must assemble all the element equations. In other words we must combine local element equations for all elements used for discretization. Element connectivities are used for the assembly process. Before solution

Boundary condition (which are not accounted in element equations) should be imposed. The fifth step is to solve the global equation system. The finite element global equation system is typically sparse, symmetric and positive definite. The sixth step is to compute additional results. In many cases we need to calculate additional parameters, for example, in mechanical problems strains and stresses are of interest in addition to displacements, which are obtained after solution of the global equation system.

The method was first developed in 1956 for the analysis of aircraft structural problems. Thereafter, within a decade, the potentialities of the method for the solution of different types of applied science and engineering problems were recognized over the years, the finite element technique has been so well established that today it is considered to be one of the best method for solving a wide variety of practical problems efficiently. In fact, the method has become one of

the active research areas for applied mathematicians. One of the main reasons for the popularity of the method in different fields of engineering is that once a general computer program is written it can be used for the solution of any problem simply by changing the input data. Now we will talk about our main subject. The superconvergence of finite element solutions is an interesting and useful phenomenon in scientific computing of real world problems. In recent years, superconvergence for finite element solutions has been an active research area in numerical analysis. The main objective in the superconvergence study is to improve the existing approximation accuracy by applying certain post processing techniques that are easy to implement. The superconvergence is obtained by applying the L^2 -projection method for the finite element approximations and their close relatives. In this article, we consider the poisson problem and shall develop a general superconvergence results for it is conforming finite element approximation.

The paper is organized as follows. In section 2, we present review for the elliptic problem and the method to find a linear system. In section 3, we develop a general theory of superconvergence result by following the idea presented in Wang [3]. In section 4, several numerical experiments are present to support the theoretical result.

2. PRELIMINARIES AND NOTATION

Let us consider the poisson problem with Robin boundary condition. Let Ω be bounded and open subset of \mathbb{R}^2 with Lipschitz continuous boundary $\partial\Omega$.

The poisson problem seeks an known function u satisfying

$$-\Delta u = f \text{ in } \Omega \quad (2.1)$$

$$\frac{\partial u}{\partial n} + u = g \text{ in } \partial\Omega \quad (2.2)$$

Where f and g are given function. Here n is the outward normal vector and $\frac{\partial u}{\partial n} = \nabla u \cdot n$ where ∇u is the gradient of u and Δ is the Laplacian operator.

Let $L^2(\Omega)$ be the space of square integrable function define on Ω with inner product and norm

$$\|u\| = \left(\int_{\Omega} |v|^2 dx \right)^{\frac{1}{2}}.$$

Further, let $H^m(\Omega)$ be Hilbert Sobolev space of order m , that is

$$H^m(\Omega) = \{v \in L^2(\Omega) : D^{\alpha} v \in L^2(\Omega), |\alpha| \leq m\} \text{ With inner product}$$

$$(u, v)_{H^m} = \sum_{|\alpha| \leq m} \int_{\Omega} D^{\alpha} u D^{\alpha} v dx \text{ and norm}$$

$$\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leq m} \int_{\Omega} |D^{\alpha} u|^2 dx \right)^{\frac{1}{2}}$$

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The weak formulation of (2.1)-(2.2) is to seek $u \in H^1(\Omega)$ such that

$$a(u, v) = (f, v)_\Omega + (g, v)_{\partial\Omega} \quad \forall v \in H^1(\Omega) \quad (2.3)$$

Where

$$a(u, v) = (\nabla u, \nabla v)_\Omega + (u, v)_{\partial\Omega} = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} u \cdot v \, ds \quad (2.4)$$

Let T_h be a finite element partition of the domain Ω with characteristic mesh size h and $p_r(k)$ be the set of all the polynomials defined on k with degree less or equal to r . Define

$$V_h = \{ v \in H^1(\Omega), v|_k \in P_1(k), \forall k \in T_h \}$$

The finite element approximation problem for (2.1)-(2.2) is to find $u_h \in V_h$ such that

$$a(u_h, v) = (f, v)_\Omega + (g, v)_{\partial\Omega} \quad \forall v \in V_h \quad (2.5)$$

Where

$$a(u_h, v) = (\nabla u_h, \nabla v)_\Omega + (u_h, v)_{\partial\Omega} = \int_\Omega \nabla u_h \cdot \nabla v \, dx + \int_{\partial\Omega} u_h \cdot v \, ds \quad (2.6)$$

Note that V_h satisfies the following approximation property.

$$\inf_{x \in V_h} (\|v - x\| + h\|v - x\|_1) \leq Ch^{r+1} \|v\|_{H^{r+1}(\Omega)}, \quad v \in H^{r+1}(\Omega) \quad (2.7)$$

Define a norm $\|\cdot\|_1$ for $V = H^1(\Omega)$ as follows:

$$\|v\|_1^2 = a(v, v) = (\nabla v, \nabla v)_\Omega + (v, v)_{\partial\Omega} = \int_\Omega \nabla v \cdot \nabla v \, dx + \int_{\partial\Omega} v^2 \, ds \quad (2.8)$$

Define the basis function $\varphi_i \in V_h$ with $\varphi_i(x_j) = 1$ where $i=j$ and $\varphi_i(x_j) = 0$ otherwise.

Our approximate solution u_h can be written in terms of its expansion coefficients and the basis function as

$$u_h(x) = \sum_{i=1}^n c_i \varphi_i(x_i)$$

Expand in basis u_h insert into (2.5) and set $v = \varphi_i, i = 1, \dots, n$

$$\sum_{i=1}^n c_i (\nabla \varphi_i, \nabla \varphi_j)_\Omega + \sum_{i=1}^n c_i (\varphi_i, \varphi_j)_{\partial\Omega} = (f, \varphi_i)_\Omega + (g, \varphi_i)_{\partial\Omega}$$

This is a linear system of equations $Ax=b$ with $A = (a_{ij}), x = (c_i), b = (b_i)$, for $i, j = 1, \dots, n$

$$a_{ij} = (\nabla \varphi_i, \nabla \varphi_j)_{\Omega} + (\varphi_i, \varphi_j)_{\partial \Omega}$$

$$b_i = (f, \varphi_i)_{\Omega} + (g, \varphi_i)_{\partial \Omega}$$

Example 2.1: Let the domain $\Omega = [0,1] \times [0,1]$. Also let the exact solution is assumed to be

$$u = \sin(\pi x) \cos(\pi y)$$

$$-\Delta u = 2\pi^2 \sin(\pi x) \cos(\pi y) \quad \text{in } \Omega$$

$$\frac{\partial u}{\partial n} + u = g \quad \text{in } \partial \Omega$$

To find the linear system $Ax = b$ see Figure 1

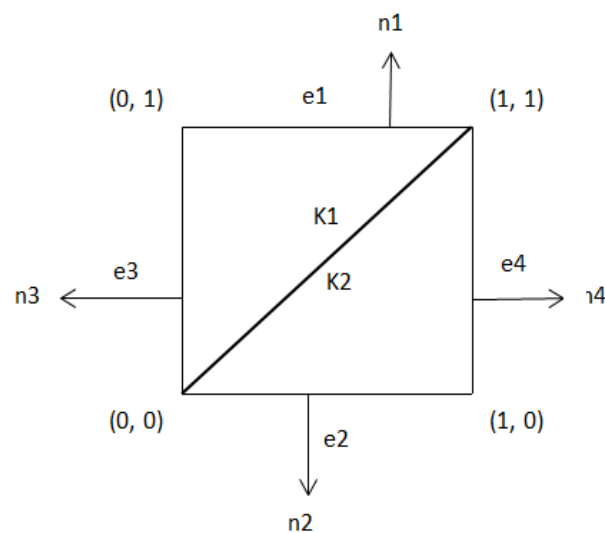


Figure 1

$$a_{ij} = (\nabla \varphi_i, \nabla \varphi_j)_{\Omega} + (\varphi_i, \varphi_j)_{\partial \Omega}$$

$$b_i = (f, \varphi_i)_{\Omega} + (g, \varphi_i)_{\partial \Omega} \quad , \quad i, j = 1, \dots, 4$$

For k_1 :

$$\varphi_1 = y - x, \text{ then } \nabla \varphi_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\varphi_3 = 1 - y, \text{ then } \nabla \varphi_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\varphi_2 = x, \text{ then } \nabla \varphi_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For k_2 :

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$$\varphi_4 = x - y, \quad \text{then } \nabla \varphi_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\varphi_3 = 1 - x, \quad \text{then } \nabla \varphi_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\varphi_2 = y, \quad \text{then } \nabla \varphi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$(\nabla \varphi_1, \nabla \varphi_1) = \int_{k_1} \nabla \varphi_1 \cdot \nabla \varphi_1 \, dk_1 = \frac{1}{2} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 1$$

$$(\nabla \varphi_2, \nabla \varphi_2) = \int_{k_1} \nabla \varphi_2 \cdot \nabla \varphi_2 \, dk_1 + \int_{k_2} \nabla \varphi_2 \cdot \nabla \varphi_2 \, dk_2$$

$$= \frac{1}{2} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$$

Then

$$A_1 = \begin{bmatrix} 1 & \frac{-1}{2} & \frac{-1}{2} & 0 \\ \frac{-1}{2} & 1 & 0 & \frac{-1}{2} \\ \frac{-1}{2} & 0 & 1 & \frac{-1}{2} \\ 0 & \frac{-1}{2} & \frac{-1}{2} & 1 \end{bmatrix}$$

$$\begin{aligned} (\varphi_1, \varphi_1)_{\partial\Omega} &= \int_{e_1} \varphi_1 \cdot \varphi_1 \, ds + \int_{e_2} \varphi_1 \cdot \varphi_1 \, ds \\ &= \int_0^1 (1-x)^2 \, dx + \int_0^1 y^2 \, dy = \frac{2}{3} \end{aligned}$$

$$(\varphi_1, \varphi_2)_{\partial\Omega} = \int_{e_1} \varphi_1 \cdot \varphi_2 \, ds = \int_0^1 x(1-x) \, dx = \frac{1}{6}$$

Then

$$A_2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{2}{3} & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{2}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{bmatrix}$$

Since $A = A_1 + A_2$

Then

$$A = \begin{bmatrix} \frac{5}{3} & \frac{-1}{3} & \frac{-1}{3} & 0 \\ \frac{-1}{3} & \frac{5}{3} & 0 & \frac{-1}{3} \\ \frac{-1}{3} & 0 & \frac{5}{3} & \frac{-1}{3} \\ 0 & \frac{-1}{3} & \frac{-1}{3} & \frac{5}{3} \end{bmatrix}$$

Now to find the load vector

$$\nabla u = \begin{bmatrix} \pi \cos(\pi x) \cos(\pi y) \\ -\pi \sin(\pi x) \sin(\pi y) \end{bmatrix}$$

Then $f = 2\pi^2 \sin(\pi x) \cos(\pi y)$

$$b_1^* = (f, \varphi_1) = \int_{k_1} f \varphi_1 dk_1 = \int_0^1 \int_x^1 2\pi^2 \sin(\pi x) \cos(\pi y) (y - x) dy dx$$

$$= \frac{-4}{\pi}$$

$$b_2^* = (f, \varphi_2) = \int_{k_1} f \varphi_2 dk_1 + \int_{k_2} f \varphi_2 dk_2 = \frac{-\pi}{2} + \frac{-8 + \pi^2}{2\pi}$$

$$b^* = \begin{bmatrix} \frac{-4}{\pi} \\ -\pi \\ \frac{-8 + \pi^2}{2} + \frac{2\pi}{8 - \pi^2} + \frac{\pi}{2\pi} \\ \frac{4}{\pi} \end{bmatrix}$$

$$g_1(x, y) = \nabla u \cdot \vec{n}_1 + u = (-\pi \sin(\pi y) + \cos(\pi y) \sin(\pi x))$$

$$g_2(x, y) = \nabla u \cdot \vec{n}_2 + u = \sin(\pi x) (\pi \sin(\pi y) + \cos(\pi y))$$

$$g_3(x, y) = \nabla u \cdot \vec{n}_3 + u = \cos(\pi y) (\sin(\pi x) - \pi \cos(\pi x))$$

$$g_4(x, y) = \nabla u \cdot \vec{n}_4 + u = \cos(\pi y) (\pi \cos(\pi x) + \sin(\pi x))$$

$$\begin{aligned} b_1^\wedge &= (g, \varphi_1)_{\partial\Omega} = \int_{e_1} g \varphi_1 ds + \int_{e_2} g_3 \varphi_1 ds \\ &= \int_0^1 -\sin(\pi x)(1-x) dx + \int_0^1 -\pi \cos(\pi y) dy = \frac{-1}{\pi} + \frac{2}{\pi} = \frac{1}{\pi} \end{aligned}$$

$$\text{Then } b^\wedge = \begin{bmatrix} \frac{1}{\pi} \\ \frac{1}{\pi} \\ \frac{1}{\pi} \\ \frac{1}{\pi} \end{bmatrix}$$

$$Ax = b, \quad \text{then } x = A^{-1}b$$

Lemma 2.1

There exists a constant, C , independent of h such th

$$|a(v, w)| \leq C \|v\|_1 \|w\|_1, \quad \forall w, v \in H^1(\Omega)$$

Proof: since the problem (2.1) – (2.2) satisfy the weak form

$$a(w, v) = (f, v)_\Omega + (g, v)_{\partial\Omega} \quad \forall v \in H^1(\Omega)$$

Where

$$a(w, v) = (\nabla w, \nabla v)_{\Omega} + (w, v)_{\partial\Omega} = \int_{\Omega} \nabla w \cdot \nabla v \, dx + \int_{\partial\Omega} w \cdot v \, ds$$

By using Schwarz inequality we obtain

$$\begin{aligned} |a(w, v)| &= \left| \int_{\Omega} \nabla w \cdot \nabla v \, dx \right| + \left| \int_{\partial\Omega} w \cdot v \, ds \right| \\ &\leq \|\nabla w\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|w\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} \end{aligned}$$

By the trace inequality there is a constant, C , such that

$$\|v\|_{L^2(\partial\Omega)} \leq C \|v\|_1$$

We have

$$\begin{aligned} |a(w, v)| &\leq \|\nabla w\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + C \|w\|_1 C \|v\|_1 \\ &\leq \|\nabla w\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + C^2 \|w\|_1 \|v\|_1 \end{aligned}$$

$$\text{Since } \|\nabla w\|_{L^2(\Omega)} \leq \|w\|_1$$

$$\text{Where } \|\nabla w\|_{L^2(\Omega)} = \left(\int_{\Omega} |\nabla w|^2 \, dx \right)^{\frac{1}{2}}, \|w\|_1 = \left(\int_{\Omega} |\nabla w|^2 \, dx + \int_{\partial\Omega} w^2 \, ds \right)^{\frac{1}{2}}$$

Then we have

$$\begin{aligned} |a(w, v)| &\leq \|w\|_1 \|v\|_1 + C^2 \|w\|_1 \|v\|_1 \\ &\leq (1 + C^2) \|w\|_1 \|v\|_1 \\ &\leq C \|w\|_1 \|v\|_1 \end{aligned}$$

$$\text{Where } C = 1 + C^2$$

And thus the proof is complete. ■

Theorem 2.1

Let u and u_h be the solution of (2.1)-(2.5) respectively. Then, there exist a constant, C , independent of h such that

$$\|u - u_h\|_1 \leq Ch^k \|u\|_{k+1}$$

Proof:

Since

$$\|u - u_h\|_1 \leq \inf_{v \in V_h} \|u - v\|_1$$

and by using (2.7) then

$$\|u - u_h\|_1 \leq Ch^{k+1} \|u\|_{k+1} \text{ for } u \in H^{k+1}(\Omega) \quad \blacksquare$$

3. SUPERCONVERGENCE BY L^2 -PROJECTION

The L^2 -projection technique was introduced by Wang [3]. It projects the approximate solution u_h to another finite element dimensional space associated with a coarse mesh. The difference in size of the two meshes can be used to achieve a superconvergence. Now, we start with defining a coarse mesh \mathcal{T}_τ where $\tau \gg h$ satisfying:

$$\tau = h^\alpha \quad (3.1)$$

With $\alpha \in (0, 1)$. For now, the parameter α will play an important role in the post processing. Define a finite element space $V_\tau \subset H^{s-2}(\Omega)$.

Here V_τ consists of piecewise polynomials of degree r associated with the partition \mathcal{T}_τ .

Let Q_τ be the L^2 -projector onto finite element space V_τ . Q_τ can be considered a linear operator (projection) from $L^2(\Omega)$ onto the finite element space V_τ . We will assume an H^s , $s \geq 1$, regularity for the solution of (2.1), (2.2):

$$\|u\|_s \leq C \|f\|_{s-2} \quad (3.2)$$

Lemma 3.1

Suppose that (3.2) holds with $1 \leq s \leq k+1$ and the finite element space $V_\tau \subset H^{s-2}(\Omega)$. Then, there exists a constant C , independent of h and τ , such that

$$\|Q_\tau u - Q_\tau u_h\| \leq Ch^{s-1+\alpha \min(0, 2-s)} \|u - u_h\|_1, \quad (3.3)$$

Where $\alpha \in (0, 1)$ is the parameter defined in (3.1).

Proof: The definition of $\|\cdot\|$ and Q_τ give

$$\|Q_\tau u - Q_\tau u_h\| = \sup_{\varphi \in L^2(\Omega), \|\varphi\|=1} \|(Q_\tau u - Q_\tau u_h, \varphi)\|$$

From definition of the L^2 -projection for Q_τ , we have

$$(Q_\tau u - Q_\tau u_h, \varphi) = (u - u_h, Q_\tau \varphi).$$

Then

$$\|Q_\tau u - Q_\tau u_h\| = \sup_{\varphi \in L^2(\Omega), \|\varphi\|=1} \|(u - u_h, Q_\tau \varphi)\|, \quad (3.4)$$

Consider the following problem: find $w \in H^1(\Omega)$ such that

$$a(w, v) = (Q_\tau \varphi, v) \quad \forall v \in H^1(\Omega) \quad (3.5)$$

It's follows from (3.2), $w \in H^s(\Omega) \cap H^1(\Omega)$ and

$$\|w\|_s \leq \|Q_\tau \varphi\|_{s-2}, \quad (3.6)$$

For some constant C . Using the inverse inequality on the right hand side of (3.6), we have.

$$\|w\|_s \leq C \tau^{\min(0, 2-s)} \|\varphi\|_0 \quad (3.7)$$

Subtracting (2.5) from (2.3) gives

$$a(u - u_h, v) = 0, \quad \forall v \in V_h \quad (3.8)$$

By replacing v by $u - u_h$ in (3.5) and using (3.8), we obtain

$$\begin{aligned} (u - u_h, Q_\tau \varphi) &= a(u - u_h, w) \\ &= a(u - u_h, w - v) \end{aligned} \quad (3.9)$$

Where $v \in V_h$. Using Lemma (2.1) and Theorem (2.1) we obtain from (3.9)

$$\begin{aligned} |(u - u_h, Q_\tau \varphi)| &= |a(u - u_h, w - v)| \\ &\leq C \inf_{v \in V_h} \|u - u_h\|_1 \|w - v\|_1 \\ &\leq Ch^{s-1} \|u - u_h\|_1 \|w\|_s. \end{aligned} \quad (3.10)$$

From (3.6), (3.7) and (3.1) we obtain

$$\begin{aligned} |(u - u_h, Q_\tau \varphi)| &\leq Ch^{s-1} \|Q_\tau \varphi\|_{s-2} \|u - u_h\|_1 \\ &\leq Ch^{s-1} \tau^{\min(0, 2-s)} \|u - u_h\|_1 \|\varphi\| \\ &\leq Ch^{s-1+\min(0, 2-s)} \|u - u_h\|_1. \end{aligned}$$

Then

$$\|Q_\tau u - Q_\tau u_h\| \leq Ch^{s-1+\min(0, 2-s)} \|u - u_h\|_1. \blacksquare$$

Theorem 3.1

Assume that (3.2) with $1 \leq s \leq k+1$ and the finite element space $V_\tau \subset H^{\delta-2}(\Omega)$. If the exact solution $u \in H^{k+1}(\Omega) \cap H^{r+1}(\Omega) \cap H^1(\Omega)$, then there exists a constant, C , such that

$$\begin{aligned} & \|u - Q_\tau u_h\| + h^\alpha \|\nabla_\tau(u - Q_\tau u_h)\| \\ & \leq Ch^{\alpha(r+1)} \|u\|_{r+1} + Ch^\sigma \|u - u_h\|_1 \end{aligned} \quad (3.10)$$

Where $\sigma = s - 1 + \alpha \min(0, 2-s)$ and u_h is the finite element approximation of the solution u .

Proof: Since Q_τ is the L^2 -projection, then

$$\|u - Q_\tau u\| \leq C\tau^{r+1} \|u\|_{r+1} \leq Ch^{\alpha(r+1)} \|u\|_{r+1}. \quad (3.11)$$

Since

$$\|u - Q_\tau u_h\| \leq \|u - Q_\tau u\| + \|Q_\tau u - Q_\tau u_h\|$$

Combining (3.3) and (3.11) in above inequality gives

$$\|u - Q_\tau u_h\| \leq Ch^{\alpha(r+1)} \|u\|_{r+1} + Ch^{s-1+\alpha \min(0, 2-s)} \|u - u_h\|_1,$$

Which complete the estimate for $\|u - Q_\tau u_h\|$ in (3.10)

Now we estimate $\|\nabla_\tau(u - Q_\tau u_h)\|$ as follows. Since, Q_τ is L^2 -projection and using (3.1) we have

$$\|\nabla_\tau(u - Q_\tau u_h)\| \leq C\tau^r \|u\|_{r+1} \leq Ch^{\alpha r} \|u\|_{r+1}. \quad (3.12)$$

Multiplying both sides of (3.12) by h^α , we get

$$h^\alpha \|\nabla_\tau(u - Q_\tau u_h)\| \leq Ch^{\alpha(r+1)} \|u\|_{r+1} \quad (3.13)$$

The inverse inequality gives:

$$\|\nabla_\tau(Q_\tau u - Q_\tau u_h)\| \leq C\tau^{-1} \|Q_\tau u - Q_\tau u_h\| \quad (3.14)$$

Using (3.1) and multiplying (3.14) by h^α the above inequality implies

$$\begin{aligned} & h^\alpha \|\nabla_\tau(Q_\tau u - Q_\tau u_h)\| \leq C \|Q_\tau u - Q_\tau u_h\| \\ & \leq Ch^{s-1+\alpha \min(0, 2-s)} \|u - u_h\|_1, \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} & h^\alpha \|\nabla_\tau(u - Q_\tau u_h)\| \leq h^\alpha \|\nabla_\tau(u - Q_\tau u)\| + h^\alpha \|\nabla_\tau(Q_\tau u - Q_\tau u_h)\| \\ & \leq Ch^{\alpha(r+1)} \|u\|_{r+1} + Ch^{s-1+\alpha \min(0, 2-s)} \|u - u_h\|_1 \end{aligned}$$

Then

$$\| (u - Q_\tau u_h) \| + h^\alpha \| \nabla_\tau (u - Q_\tau u_h) \| \leq Ch^{\alpha(r+1)} \| u \|_{r+1} + Ch^{k+s-1+\alpha \min(0,2-s)} \| u \|_{k+1}$$

The above error estimate is optimized if α is selected so that

$$\alpha(r+1) = k + s - 1 + \alpha \min(0, 2 - s).$$

Solving α from above yields

$$\alpha = \frac{k+s-1}{r+1-\min(0,2-s)} \quad (3.16)$$

The corresponding error estimate is given by

$$\| (u - Q_\tau u_h) \| + h^\alpha \| \nabla_\tau (u - Q_\tau u_h) \| \leq Ch^{\alpha(r+1)} (\| u \|_{r+1} + \| u \|_{k+1}) \quad (3.17)$$

Theorem 3.2

Suppose that (3.2) holds with $1 \leq s \leq k+1$ and the finite element space $V_\tau \in H^{s-2}(\Omega)$. Let u_h be the finite element approximation of the solution. Then the post-processed of u is estimated by (3.17) with

$$\alpha = \frac{k+s-1}{r+1-\min(0,2-s)}$$

4. NUMERICAL EXAMPLES FOR CONFORMING FEM BY L^2 -PROJECTION METHOD

In this section, we present several numerical examples to verify Theorem 3.3. The triangulation of T_h is constructed by: (1) dividing the domain into an $n^3 \times n^3$ rectangular mesh and (2) connecting the diagonal line with the positive slope. Denote $h = \frac{1}{n^3}$ as the mesh size. The finite element spaces are defined by

$$V_h = \{v \in H^1(\Omega) : v|_k \in P_1(k), \forall k \in T_h\}$$

$$V_\tau = \{v \in L^2(\Omega) : v|_k \in P_2(K), \forall k \in T_\tau\}$$

Example 4.1: Let the domain $\Omega = [0,1] \times [0,1]$ and the exact solution is assumed as

$$u = (x^3 - x^2)(y^3 - y^2)$$

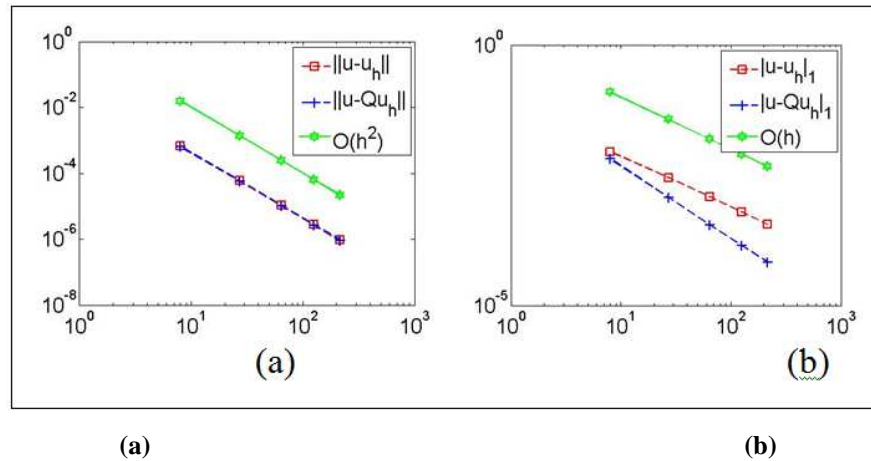
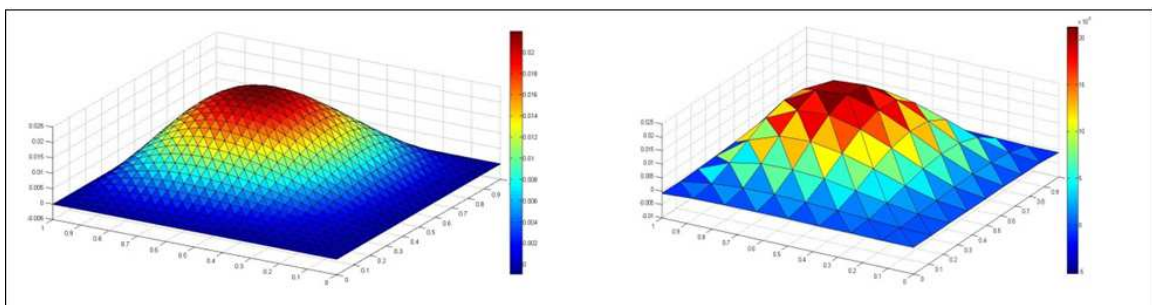
Table 1 shows that after post-processing method, all the errors are reduced. The exact solution in L^2 -norm of $\|u - Q_\tau u_h\|_1$ has the similar convergence

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Rate as $\|u - u_h\|$. There is no improvement for the u in L^2 -norm. However, the error in the H^1 -norm has a higher convergence rate, which is shown as $O(h^{1.3})$ for $\|\nabla_\tau(u - Q_\tau u_h)\|$, see Figure 2(a) and (b). Figure 3(a) and (b) give the results for the finite element approximation of (2.1) and (2.2) before and after post-processing.

Table 1: Error on Uniform Triangular Meshes T_h And T_τ

h	$\ u - u_h\ _1$	$\ u - Q_\tau u_h\ _1$	$\ u - Q_\tau u_h\ $
2^{-3}	0.8936e-2	0.6620e-3	0.6455e-2
3^{-3}	0.2842e-2	0.5986e-4	0.1172e-2
4^{-3}	0.1212e-2	0.1066e-4	0.3510e-3
5^{-3}	0.6224e-3	0.2795e-5	0.1404e-3
6^{-3}	0.3604e-3	0.9358e-6	0.6698e-4
$o(h^k), k=$	0.9752	1.9915	1.3873

Figure 2: (a) Convergence Rate of L^2 -Norm Error; (b) Convergence Rate of H^1 -Norm ErrorFigure 3: Result for $u = (x^3 - x^2)(y^3 - y^2)$ In Example 4.1. (a) Surface Plot of Approximation Solution u_h ; (b) Surface Plot of Approximation Solution $Q_\tau u_h$

Example 4.2: Let the domain $\Omega = [0,1] \times [0,1]$ and the exact solution is assumed as

$$u = x^5 \cos(y)$$

From the results shown in Table 2, it is clear that the exact solution u in H^1 -norm has the superconvergence but

there is no improvement in L^2 -norm, see Figures 4 (a) and (b). The finite element solution given in Figures 5 (a) and (b). This agrees well with the theory.

Table 2: Error on Uniform Triangular Meshes T_h and T_τ

h	$\ u - u_h\ _1$	$\ u - u_h\ $	$\ u - Q_\tau u_h\ _1$	$\ u - Q_\tau u_h\ $
3^{-3}	0.3407e-10	5.738e-3	0.8608e-20	4.887e-3
4^{-3}	0.1441e-10	1.023e-3	0.2618e-20	8.709e-4
5^{-3}	0.7384e-20	2.684e-40	1.065e-20	2.288e-4
6^{-3}	0.4274e-20	8.991e-50	5.135e-30	7.648e-5
$O(h^k), k =$	0.99821	0.99861	3.5561	0.9993

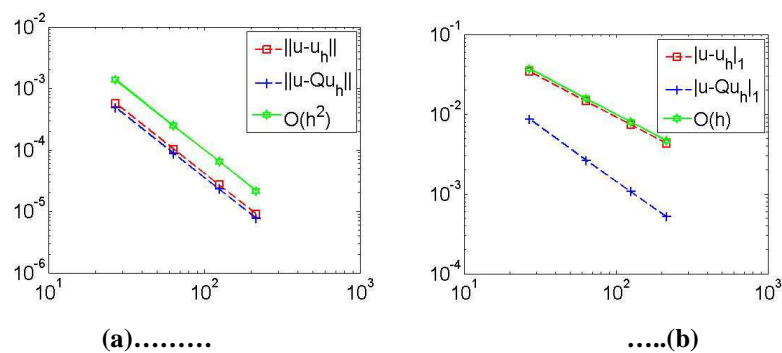


Figure 4: (a) Convergence Rate of L^2 -Norm Error; (b) Convergence Rate of H^1 -Norm Error

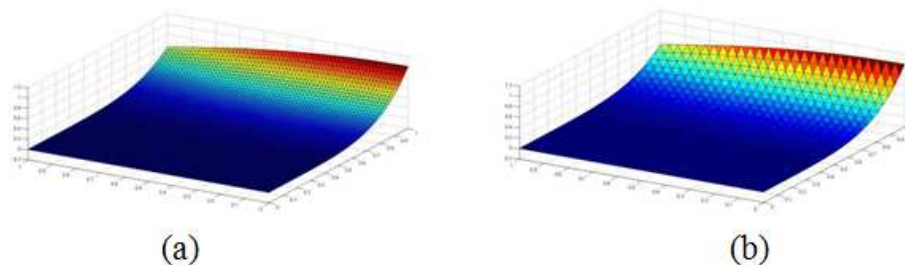


Figure 5: Result for $u = x^5 \cos(y)$ in Example 4.1 (a) Surface Plot of Approximation Solution u_h ; (b) Surface Plot of Approximation Solution $Q_\tau u_h$

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